Solid Mechanics - 202041

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Unit IV Torsion, Buckling

Torsion of circular shafts: Introduction to torsion on a shaft with application, Basic torsion formulae and assumption in torsion theory, Torsion in stepped and composite shafts, Torque transmission on strength and rigidity basis, Torsional Resilience Torsion on Thin-Walled Tubes: Introduction of Torsion on Thin-Walled Tubes Shaft and its application

Buckling of columns: Introduction to buckling of column with its application, Different column conditions and critical, safe load determination by Euler's theory. Limitation of Euler's theory

Introduction

In workshops and factories, a turning force is always applied to transmit energy by rotation. This turning force is applied either to the rim of a pulley, keyed to the shaft or at any other suitable point at some distance from the axis of the shaft. The product of this turning force and the distance between the point of application of the force and the axis of the shaft is known as torque, turning moment or twisting moment. And the shaft is said to be subjected to torsion. Due to this torque, every cross-section of the shaft is subjected to some shear stress.

Assumptions For Shear Stress in Circular Shaft Subjected to Torsion

1. The material of the shaft is uniform throughout.
2. The twist along the shaft is uniform.
3. Normal cross-sections of the shaft, which were plane and circular before the twist, remain plane and circular even after the twist.
4. All diameters of the normal cross-section, which were straight before the twist, remain straight with their magnitude unchanged, after the twist.

A little consideration will show that the above assumptions are justified, if the torque applied is small and the angle of twist is also small.

Torsional Stresses and Strains

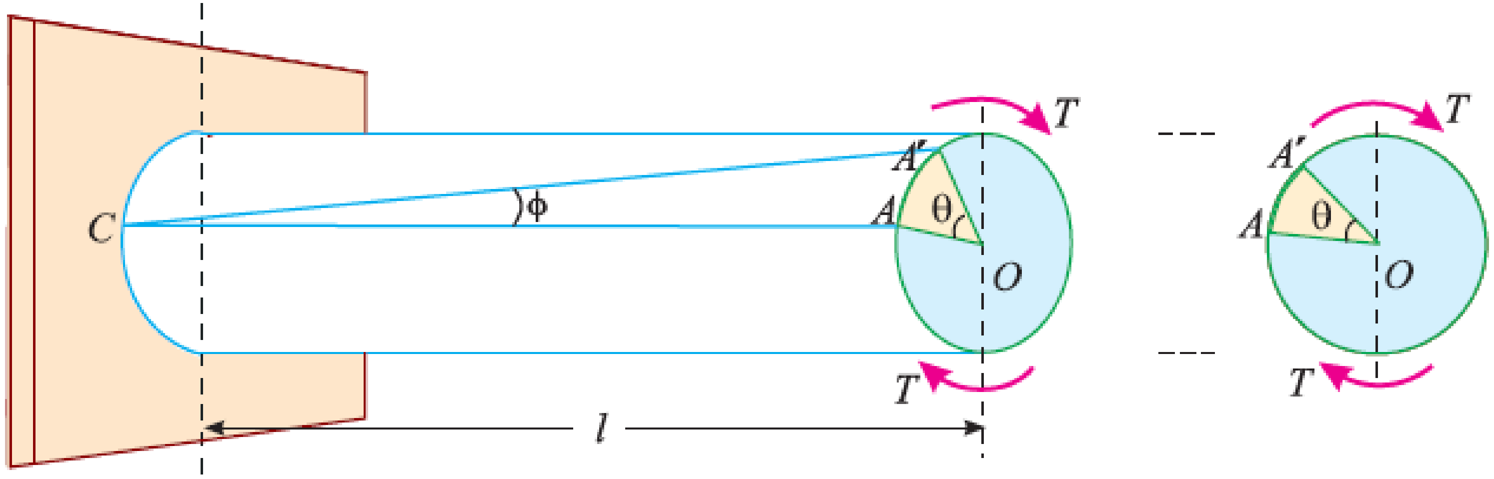


Fig. 27.1

Consider a circular shaft fixed at one end and subjected to a torque at the other end as shown in Fig. 27.1.

Let T = Torque in N-mm,

l = Length of the shaft in mm and

R = Radius of the circular shaft in mm.

As a result of this torque, every cross-section of the shaft will be subjected to shear stresses. Let the line CA on the surface of the shaft be deformed to CA' and OA to OA' as shown in Fig. 27.1.

Let $\angle ACA' = \phi$ in degrees

$\angle AOA' = \theta$ in radians

τ = Shear stress induced at the surface and

C = Modulus of rigidity, also known as torsional rigidity of the shaft material.

We know that shear strain = Deformation per unit length

$$= \frac{AA'}{l} = \tan\theta$$

$$= \phi \qquad \text{...(}\phi \text{ being very small, } \tan\phi = \phi\text{)}$$

We also know that the arc $AA' = R \cdot \theta$

$$\therefore \quad \phi = \frac{AA'}{l} = \frac{R \cdot \theta}{l} \qquad \text{...}(i)$$

If τ is the intensity of shear stress on the outermost layer and C the modulus of rigidity of the shaft, then

$$\phi = \frac{\tau}{C} \qquad \text{...}(ii)$$

From equations (*i*) and (*ii*), we find that

$$\frac{\tau}{C} = \frac{R \cdot \theta}{l} \quad \text{or} \quad \frac{\tau}{R} = \frac{C \cdot \theta}{l}$$

If τ_X be the intensity of shear stress, on any layer at a distance *x* from the centre of the shaft, then

$$\frac{\tau_X}{x} = \frac{\tau}{R} = \frac{C \cdot \theta}{l} \qquad ...(iii)$$

Strength of a Solid Shaft

The term, strength of a shaft means the maximum torque or power, it can transmit. As a matter of fact, we are always interested in calculating the torque, a shaft can withstand or transmit.

Let R = Radius of the shaft in mm and

τ = Shear stress developed in the outermost layer of the shaft in N/mm^2

Consider a shaft subjected to a torque T as shown in Fig. 27.2. Now let us consider an element of area da of thickness dx at a distance x from the centre of the shaft as shown in Fig. 27.2.

$$\therefore \quad da = 2\pi x \cdot dx \quad \ldots(i)$$

and shear stress at this section,

$$\therefore \quad \tau_X = \tau \times \frac{x}{R} \quad \ldots(ii)$$

where τ = Maximum shear stress.

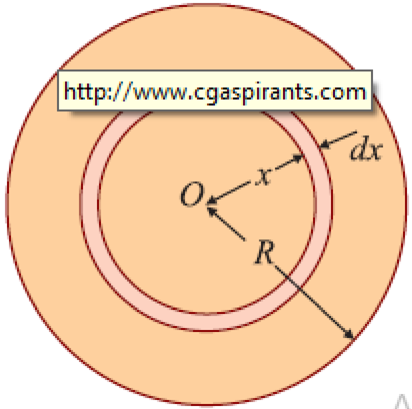
$$\therefore \quad \text{Turning force} = \text{Shear Stress} \times \text{Area}$$

$$= \tau_x \cdot dx$$

$$= \tau \times \frac{x}{R} \times da$$

$$= \tau \frac{x}{R} \times 2\pi x \cdot dx$$

$$= \frac{2\pi x}{R} \cdot x^2\, dx$$



Activate

We know that turning moment of this element,

$$dT = \text{Turning force} \times \text{Distance of element from axis of the shaft}$$

$$= \frac{2\pi\tau}{R} x^2\, dx \cdot x = \frac{2\pi\tau}{R} x^3 \cdot dx \qquad \ldots(iii)$$

The total torque, which the shaft can withstand, may be found out by integrating the above equation between 0 and R *i.e.*,

$$T = \int_0^R \frac{2\pi\tau}{R} x^3 \cdot dx = \frac{2\pi\tau}{R} \int_0^R x^2 \cdot dx$$

$$= \frac{2\pi\tau}{R} \left[\frac{x^4}{4} \right]_0^R = \frac{\pi}{2} \tau \cdot R^3 = \frac{\pi}{16} \times \tau \times D^3 \quad \text{N-mm}$$

where D is the diameter of the shaft and is equal to $2R$.

A circular shaft of 50 mm diameter is required to transmit torque from one shaft to another. Find the safe torque, which the shaft can transmit, if the shear stress is not to exceed 40 MPa.

SOLUTION. Given: Diameter of shaft (D) = 50 mm and maximum shear stress (τ) = 40 MPa = 40 N/mm^2.

We know that the safe torque, which the shaft can transmit,

$$T = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times 40 \times (50)^3 \text{ N-mm}$$

$$= 0.982 \times 10^6 \text{ N-mm} = \mathbf{0.982\ kN\text{-}m} \qquad \textbf{Ans.}$$

A solid steel shaft is to transmit a torque of 10 kN-m. If the shearing stress is not to exceed 45 MPa, find the minimum diameter of the shaft.

SOLUTION. Given: Torque (T) = 10 kN-m = 10×10^6 N-mm and maximum shearing stress (τ) = 45 MPa = 45 N/mm^2.

Let D = Minimum diameter of the shaft in mm.

We know that torque transmitted by the shaft (T),

$$10 \times 10^6 = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times 45 \times D^3 = 8.836\, D^3$$

$$\therefore \quad D^3 = \frac{10 \times 10^6}{8.836} = 1.132 \times 10^6$$

or $D = 1.04 \times 10^2 = \mathbf{104\ mm}$ **Ans.**

Power Transmitted by a Shaft

We have already discussed that the main purpose of a shaft is to transmit power from one shaft to another in factories and workshops. Now consider a rotating shaft, which transmits power from one of its ends to another.

Let N = No. of revolutions per minute and

T = Average torque in kN-m.

$$\text{Work done per minute} = \text{Force} \times \text{Distance} = T \times 2\pi N = 2\pi NT$$

$$\text{Work done per second} = \frac{2\pi NT}{60} \text{ kN-m}$$

$$\text{Power transmitted} = \text{Work done in kN-m per second}$$

$$= \frac{2\pi NT}{60} \text{ kW}$$

A circular shaft of 60 mm diameter is running at 150 r.p.m. If the shear stress is not to exceed 50 MPa, find the power which can be transmitted by the shaft.

Solution. Given: Diameter of the shaft (D) = 60 mm ; Speed of the shaft (N) = 150 r.p.m. and maximum shear stress (τ) = 50 MPa = 50 N/mm^2.

We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times 50 \times (60)^3 \text{ N-mm}$$

$$= 2.12 \times 10^6 \text{ N-mm} = 2.12 \text{ kN-m}$$

and power which can be transmitted by the shaft,

$$P = \frac{2\pi NT}{60} = \frac{2\pi \times 150 \times 2.12}{60} = \mathbf{33.3\ kW} \qquad \textbf{Ans.}$$

A hollow shaft of external and internal diameters as 100 mm and 40 mm is transmitting power at 120 r.p.m. Find the power the shaft can transmit, if the shearing stress is not to exceed 50 MPa.

SOLUTION. Given: External diameter (D) = 100 mm; Internal diameter (d) = 40 mm ; Speed of the shaft (N) = 120 r.p.m. and allowable shear stress (τ) = 50 MPa = 50 N/mm^2.

We know that torque the shaft can transmit,

$$T = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times 50 \times \left[\frac{(100)^4 - (40)^4}{100}\right] \text{ N-mm}$$

$$= 9.56 \times 10^6 \text{ N-mm} = 9.56 \text{ kN-m}$$

and power the shaft can transmit,

$$P = \frac{2\pi NT}{60} = \frac{2\pi \times 120 \times 9.56}{60} = \mathbf{120\ kW} \qquad \textbf{Ans.}$$

A solid circular shaft of 100 mm diameter is transmitting 120 kW at 150 r.p.m. Find the intensity of shear stress in the shaft.

Solution. Given : Diameter of the shaft (D) = 100 mm ; Power transmitted (P) = 120 kW and speed of the shaft (N) = 150 r.p.m.

Let T = Torque transmitted by the shaft, and

τ = Intensity of shear stress in the shaft.

We know that power transmitted by the shaft (P),

$$120 = \frac{2\pi NT}{60} = \frac{2\pi \times 150 \times T}{60} = 15.7\,T$$

$$\therefore \quad T = \frac{120}{15.7} = 7.64 \text{ kN-m} = 7.64 \times 10^6 \text{ N-mm}$$

We also know that torque transmitted by the shaft (T),

$$7.64 \times 10^6 = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times \tau \times (100)^3 = 0.196 \times 10^6\,\tau$$

$$\tau = \frac{7.64}{0.196} = 39 \text{ N/mm}^2 = \mathbf{39\ MPa} \qquad \textbf{Ans.}$$

A hollow shaft is to transmit 200 kW at 80 r.p.m. If the shear stress is not to exceed 60 MPa and internal diameter is 0.6 of the external diameter, find the diameters of the shaft.

SOLUTION. Given : Power (P) = 200 kW ; Speed of shaft (N) = 80 r.p.m. ; Maximum shear stress (τ) = 60 MPa = 60 N/mm^2 and internal diameter of the shaft (d) = 0.6D (where D is the external diameter in mm).

We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times 60 \times \left[\frac{D^4 - (0.6D)^4}{D}\right] \text{ N-mm}$$

$$= 10.3\, D^3 \text{ N-mm} = 10.3 \times 10^{-6}\, D^3 \text{ kN-m} \qquad \ldots(i)$$

We also know that power transmitted by the shaft (P),

$$200 = \frac{2\pi NT}{60} = \frac{2\pi \times 80 \times (10.3 \times 10^{-6} D^3)}{60} = 86.3 \times 10^{-6}\, D^3$$

$$\therefore \quad D^3 = \frac{200}{(86.3 \times 10^{-6})} = 2.32 \times 10^6 \text{ mm}^3$$

or $\quad D = 1.32 \times 10^2 = \mathbf{132\ mm}$ **Ans.**

and $\quad d = 0.6\, D = 0.6 \times 132 = \mathbf{79.2\ mm}$ **Ans.**

A solid steel shaft has to transmit 100 kW at 160 r.p.m. Taking allowable shear stress as 70 MPa, find the suitable diameter of the shaft. The maximum torque transmitted in each revolution exceeds the mean by 20%.

SOLUTION. Given: Power $(P) = 100$ kW ; Speed of the shaft $(N) = 160$ r.p.m. ; Allowable shear stress $(\tau) = 70$ MPa $= 70$ N/mm^2 and maximum torque $(T_{max}) = 1.2\,T$ (where T is the mean torque).

Let D = Diameter of the shaft in mm.

We know that power transmitted by shaft (P),

$$100 = \frac{2\pi NT}{60} = \frac{2\pi \times 160 \times T}{60} = 16.8\,T$$

$$\therefore \quad T = \frac{100}{16.8} = 5.95 \text{ kN-m} = 5.95 \times 10^6 \text{ N-mm}$$

and maximum torque, $T_{max} = 1.2T = 1.2 \times (5.95 \times 10^6) = 7.14 \times 10^6$ N-mm

We also know that maximum torque (T_{max}),

$$7.14 \times 10^6 = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times 70 \times D^3 = 13.7\,D^3$$

$$\therefore \quad D^3 = \frac{7.14 \times 10^6}{13.7} = 0.521 \times 10^6$$

or $D = 0.8 \times 10^2 =$ **80 mm** **Ans.**

Polar Moment of Inertia

The moment of inertia of a plane area, with respect to an axis perpendicular to the plane of the figure, is called polar moment of inertia with respect to the point, where the axis intersects the plane. In a circular plane, this point is always the centre of the circle. We know that

$$\frac{\tau}{R} = \frac{C \cdot \theta}{l} \quad \text{...}(i) \quad \text{... (from Art. 27.3)}$$

and

$$T = \frac{\pi}{16} \times \tau \times D^3 \quad \text{...}(ii) \quad \text{... (from Art. 27.3)}$$

or

$$\tau = \frac{16T}{\pi D^3}$$

Substituting the value of τ in equation (i),

$$\frac{16T}{\pi D^3 \times R} = \frac{C \cdot \theta}{l}$$

or

$$\frac{T}{\frac{\pi}{16} \times D^3 \times R} = \frac{C \cdot \theta}{l}$$

$$\frac{T}{\frac{\pi}{32} \times D^4} = \frac{C \cdot \theta}{l} \quad \text{...}\left(\text{Radius, } R = \frac{D}{2}\right)$$

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \qquad ...(iii)$$

where $J = \frac{\pi}{32} \times D^4$. It is known as polar moment of inertia.

The above equation (*iii*) may also be written as :

$$\frac{\tau}{R} = \frac{T}{J} = \frac{C \cdot \theta}{l} \qquad ...\left(\because \frac{\tau}{R} = \frac{C \cdot \theta}{l}\right)$$

NOTES. 1. In a hollow circular shaft the polar moment of inertia,

$$J = \frac{\pi}{32}(D^4 - d^4)$$

where d is the internal diameter of the shaft.

2. The term $\frac{J}{R}$ is known as torsional section modulus or polar modulus. It is similar to section modulus which is equal to $\frac{I}{y}$.

3. Thus polar modulus for a solid shaft,

$$Zp = \frac{2\pi}{32D} \times D^4 = \frac{\pi}{16} D^3$$

and the polar modulus for a hollow shaft,

$$Zp = \frac{2\pi}{32D}(D^4 - d^4) = \frac{\pi}{16D}(D^4 - d^4)$$

Calculate the maximum torque that a shaft of 125 mm diameter can transmit, if the maximum angle of twist is 1° in a length of 1.5 m. Take C = 70 GPa.

SOLUTION. Given: Diameter of shaft (D) = 125 mm ; Angle of twist (θ) = $1° = \frac{\pi}{180}$ rad ; Length of the shaft (l) = 1.5 m = 1.5×10^3 mm and modulus of rigidity (C) = 70 GPa = 70×10^3 N/mm^2.

Let T = Maximum torque the shaft can transmit.

We know that polar moment of inertia of a solid circular shaft,

$$J = \frac{\pi}{32} \times (D)^4 = \frac{\pi}{32} (125)^4 = 24.0 \times 10^6 \text{ mm}^4$$

and relation for torque transmitted by the shaft,

$$\frac{T}{J} = \frac{C \cdot \theta}{l}$$

$$\frac{T}{24.0 \times 10^6} = \frac{(70 \times 10^3)\pi/180}{1.5 \times 10^3} = 0.814$$

$$\therefore \quad T = 0.814 \times (24.0 \times 10^6) = 19.5 \times 10^6 \text{ N-mm}$$

$= 19.5$ **kN-m** **Ans.**

Find the angle of twist per metre length of a hollow shaft of 100 mm external and 60 mm internal diameter, if the shear stress is not to exceed 35 MPa. Take C = 85 GPa.

SOLUTION. Given: Length of the shaft (l) = 1 m = 1×10^3 mm ; External diameter (D) = 100 mm; Internal diameter (d) = 60 mm ; Maximum shear stress (τ) = 35 MPa = 35 N/mm^2 and modulus of rigidity (C) = 85 GPa = 85×10^3 N/mm^2.

Let θ = Angle of twist in the shaft.

We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times 35 \times \left[\frac{(100)^4 - (60)^4}{100}\right] \text{ N-mm}$$

$$= 5.98 \times 10^6 \text{ N-mm}$$

We also know that polar moment of inertia of a hollow circular shaft,

$$J = \frac{\pi}{32}[D^4 - d^4] = \frac{\pi}{32}[(100)^4 - (60)^4] = 8.55 \times 10^6 \text{ mm}^4$$

and relation for the angle of twist,

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \frac{5.98 \times 10^6}{8.55 \times 10^6} = \frac{(85 \times 10^3)\,\theta}{1 \times 10^3} = 85.\theta$$

$$\therefore \quad \theta = \frac{5.98 \times 10^6}{(8.55 \times 10^6) \times 85} = 0.008 \text{ rad} = \mathbf{0.5^\circ} \quad \textbf{Ans.}$$

A solid shaft of 120 mm diameter is required to transmit 200 kW at 100 r.p.m. If the angle of twists not to exceed 2°, find the length of the shaft. Take modulus of rigidity for the shaft material as 90 GPa.

SOLUTION. Given : Diameter of shaft $(D) = 120$ mm; Power $(P) = 200$ kW ; Speed of shaft $(N) = 100$ r.p.m. ; Angle of twist $(\theta) = 2° = \frac{2\pi}{180}$ rad. and modulus of rigidity $(C) = 90 \text{ GPa} = 90 \times 10^3 \text{ N/mm}^2$.

Let T = Torque transmitted by the shaft, and

l = Length of the shaft.

We know that power transmitted by the shaft (P),

$$200 = \frac{2\pi NT}{60} = \frac{2\pi \times 100 \times T}{60} = 10.5T$$

$$\therefore \quad T = \frac{200}{10.5} = 19 \text{ kN-m} = 19 \times 10^6 \text{ N-mm}$$

We also know that polar moment of inertia of a solid shaft,

$$J = \frac{\pi}{32} \times (D)^4 = \frac{\pi}{32} \times (120)^4 = 0.4 \times 10^6 \text{ mm}^4$$

and relation for the length of the shaft,

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \frac{19 \times 10^6}{20.4 \times 10^5} = \frac{(90 \times 10^3) \times (2\pi/180)}{l}$$

$$0.931 = \frac{3.14 \times 10^3}{l}$$

$$\therefore \quad l = \frac{(3.14 \times 10^3)}{0.931} = 3.37 \times 10^3 = \mathbf{3.37 \text{ m}} \quad \textbf{Ans.}$$

Find the maximum torque, that can be safely applied to a shaft of 80 mm diameter. The permissible angle of twist is 1.5 degree in a length of 5 m and shear stress not to exceed 42 MPa. Take C = 84 GPa.

SOLUTION. Given: Diameter of shaft $(D) = 80$ mm ; Angle of twist $(\theta) = 1.5° = \frac{1.5\pi}{180}$ rad ; Length of shaft $(l) = 5$ m $= 5 \times 10^3$ mm ; Maximum shear stress $(\tau) = 42$ MPa $= 42$ N/mm^2 and Modulus of rigidity $(C) = 84$ GPa $= 84 \times 10^3$ N/mm^2.

First of all, let us find out the values of torques based on shear stress and angle of twist.

1. ***Torque based on shear stress***

We know that the torque which can be applied to the shaft,

$$T_1 = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times 42 \times (80)^3 = 4.22 \times 10^6 \text{ N-mm} \quad ...(i)$$

2. ***Torque based on angle of twist***

We also know that polar moment of inertia of a solid circular shaft,

$$J = \frac{\pi}{32} (D)^4 = \frac{\pi}{32} \times (80)^4 = 4.02 \times 10^6 \text{ mm}^4$$

and relation for the torque that can be applied:

$$\frac{T_2}{J} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \frac{T_2}{4.02 \times 10^6} = \frac{(84 \times 10^3) \times (1.5\pi/180)}{5 \times 10^3} = 0.44$$

$$\therefore \quad T_2 = 0.44 \times (4.02 \times 10^6) = 1.77 \times 10^6 \text{ N-mm} \quad ...(ii)$$

We shall apply a torque of 1.77×10^6 N-mm (*i.e.*, lesser of the two values). **Ans.**

A solid shaft is subjected to a torque of 1.6 kN-m. Find the necessary diameter of the shaft, if the allowable shear stress is 60 MPa. The allowable twist is 1° for every 20 diameters length of the shaft. Take C = 80 GPa.

SOLUTION. Given: Torque $(T) = 1.6\text{ kN-m} = 1.6 \times 10^6$ N-mm; Allowable shear stress $(\tau) = 60$ MPa $=$ 60 N/mm^2; Angle of twist $(\theta) = 1° = \dfrac{\pi}{180}$ rad; Length of shaft $(l) = 20\,D$ and modulus of rigidity (C) $= 80\text{ GPa} = 80 \times 10^3\text{ N/mm}^2$.

First of all, let us find out the value of diameter of the shaft for its strength and stiffness.

1. *Diameter for strength*

We know that torque transmitted by the shaft (T),

$$1.6 \times 10^6 = \frac{\pi}{16} \times \tau \times D_1^3 = \frac{\pi}{16} \times 60 \times D_1^3 = 11.78\,D_1^3$$

$$\therefore \quad D_1^3 = \frac{1.6 \times 10^6}{11.78} = 0.136 \times 10^6\text{ mm}^3$$

or $$D_1 = 0.514 \times 10^2 = 51.4\text{ mm} \qquad ...(i)$$

2. *Diameter for stiffness*

We know that polar moment of inertia of a solid circular shaft,

$$J = \frac{\pi}{32} \times (D_2)^4 = 0.098\,D_2^4$$

and relation for the diameter,

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \frac{1.6 \times 10^6}{0.098\,D_2^4} = \frac{(80 \times 10^3) \times (\pi/180)}{20D_2}$$

$$\therefore \quad D_2^3 = \frac{(1.6 \times 10^6) \times 20}{0.098 \times (80 \times 10^3) \times (\pi/180)} = 234 \times 10^3\text{ mm}^3$$

or $$D_2 = 6.16 \times 10^1 = 61.6\text{ mm} \qquad ...(ii)$$

We shall provide a shaft of diameter of 61.6 mm (*i.e.*, greater of the two values). **Ans.**

A solid shaft of 200 mm diameter has the same cross-sectional area as a hollow shaft of the same material with inside diameter of 150 mm. Find the ratio of

(*a*) *powers transmitted by both the shafts at the same angular velocity.*

(*b*) *angles of twist in equal lengths of these shafts, when stressed to the same intensity.*

SOLUTION. Given: Diameter of solid shaft (D_1) = 200 mm and inside diameter of hollow shaft (d) = 150 mm.

(a) Ratio of powers transmitted by both the shafts

We know that cross-sectional area of the solid shaft,

$$A_1 = \frac{\pi}{4} \times D_1^2 = \frac{\pi}{4} \times (200)^2 = 10\,000\,\pi \text{ mm}^2$$

and cross-sectional area of hollow shaft,

$$A_2 = \frac{\pi}{4} \times (D^2 - d^2) = \frac{\pi}{4} \times [D^2 - (150)^2] = \frac{\pi}{4}(D^2 - 22\,500)$$

Since the cross-sectional areas of both the shafts are same, therefore equating A_1 and A_2,

$$\frac{\pi}{4}(200)^2 = \frac{\pi}{4}(D^2 - 22\,500)$$

$$\therefore \quad 40\,000 = D^2 - 22\,500$$

$$D^2 = 40\,000 + 22\,500 = 62\,500 \text{ mm}^2$$

$$\text{or} \quad D = 250 \text{ mm}$$

We also know that torque transmitted by the solid shaft,

$$T_1 = \frac{\pi}{16} \times \tau \times D_1^3 = \frac{\pi}{16} \times \tau \times (200)^3 = 500 \times 10^3 \pi \tau \text{ N-mm} \qquad ...(i)$$

Similarly, torque transmitted by the hollow shaft,

$$T_2 = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times \tau \times \left[\frac{(250)^4 - (150)^4}{250}\right] \text{ N-mm}$$

$$= 850 \times 10^3 \pi \tau \text{ N-mm}$$

$$\therefore \quad \frac{\text{Power transmitted by hollow shaft}}{\text{Power transmitted by solid shaft}}$$

$$= \frac{T_2}{T_1} = \frac{50 \times 10^3 \pi \tau}{500 \times 10^3 \pi \tau} = \mathbf{1.7} \qquad \textbf{Ans.}$$

(*b*) *Ratio of angles of twist in both the shafts*

We know that relation for angle of twist for a shaft,

$$\frac{\tau}{R} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \theta = \frac{\tau l}{RC}$$

∴ Angle of twist for the solid shaft,

$$\theta_1 = \frac{\tau l}{RC} = \frac{\tau l}{100C} \qquad ...\left(\text{where } R = \frac{D_1}{2} = \frac{200}{2} = 100 \text{ mm}\right)$$

Similarly angle of twist for the hollow shaft,

$$\theta_2 = \frac{\tau l}{RC} = \frac{\tau l}{125C} \qquad \ldots\left(\text{where } R = \frac{D_1}{2} = \frac{250}{2} = 125 \text{ mm}\right)$$

$$\therefore \quad \frac{\text{Angle of twist of hollow shaft}}{\text{Angle of twist of solid shaft}} = \frac{\theta_2}{\theta_1} = \frac{\dfrac{\tau l}{125C}}{\dfrac{\tau l}{100C}} = \frac{100}{125} = \mathbf{0.8} \qquad \textbf{Ans.}$$

Replacing a Shaft

Sometimes, we are required to replace a solid shaft by a hollow one, or vice versa. In such cases, the torque transmitted by the new shaft should be equal to that by the replaced shaft. But sometimes, there are certain other conditions which have also to be considered while designing the new shaft.

EXAMPLE 27.15. *A solid steel shaft of 60 mm diameter is to be replaced by a hollow steel shaft of the same material with internal diameter equal to half of the external diameter. Find the diametres of the hollow shaft and saving in material, if the maximum allowable shear stress is same for both shafts.*

SOLUTION. Given: Diameter of solid shaft (D) = 60 mm.

Diameter of the hollow shaft

Let D = External diameter of the hollow shaft,

d = Internal diameter of the hollow shaft (equal to $D_1/2$) and

τ = Shear stress developed in both the shafts.

We know that torque transmitted by the solid shaft,

$$T = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times \tau \times (60)^3 \quad \text{...}(i)$$

and torque transmitted by the hollow shaft,

$$T_1 = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D_1}\right] = \frac{\pi}{16} \times \tau \times \left[\frac{D_1^4 - (0.5D_1)^4}{D_1}\right]$$

$$= \frac{\pi}{16} \times \tau \times 0.9375\, D_1^3 \quad \text{...}(ii)$$

Since the torque transmitted and allowable shear stress in both the cases are same, therefore equating the equations (*i*) and (*ii*),

$$\frac{\pi}{16} \times \tau \times (60)^3 = \frac{\pi}{16} \times \tau \times 0.9375\, D_1^3$$

$$\therefore \quad D_1^3 = \frac{(60)^3}{0.9375} = 230400 \text{ mm}^3$$

or $D_1 = \mathbf{61.3\ mm}$ **Ans.**

and $d = \frac{61.3}{2} = \mathbf{30.65\ mm}$ **Ans.**

Saving in material

We know that saving in material

$$= \frac{\left[\frac{\pi}{4}(60)^2\right] - \left[\frac{\pi}{4}((61.3)^2 - (30.65)^2)\right]}{\frac{\pi}{4}(60)^2} = \frac{3600 - 2819}{3600}$$

$= 0.217 = \mathbf{21.7\%}$ **Ans.**

A solid shaft of 80 mm diameter is to be replaced by a hollow shaft of external diameter 100 mm. Determine the internal diameter of the hollow shaft if the same power is to be transmitted by both the shafts at the same angular velocity and shear stress.

SOLUTION. Given: Diameter of solid shaft $(D) = 80$ mm and external diameter of hollow shaft $(D_1) = 100$ mm.

Let d = Internal diameter of the hollow shaft, and

τ = Shear stress developed in both the shafts.

We know that torque transmitted by the solid shaft,

$$T = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times \tau \times (80)^3 \quad ...(i)$$

and torque transmitted by the hollow shaft,

$$T_1 = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times \tau \times \left[\frac{(100)^4 - d^4}{100}\right] \quad ...(ii)$$

Since both the torques are equal, therefore equating the equations (*i*) and (*ii*),

$$\frac{\pi}{16} \times \tau \times (80)^2 = \frac{\pi}{16} \times \tau \times \left[\frac{(100)^4 - d^4}{100}\right]$$

$$(80)^3 = \frac{(100)^4 - d^4}{100} = (100)^3 - \frac{d^4}{100}$$

$$\frac{d^4}{100} = (100)^3 - (80)^3 = 488 \times 10^3$$

$$d^4 = (488 \times 10^3) \times 100 = 488 \times 10^5 = 4880 \times 10^4$$

$$\therefore \quad d = 8.36 \times 10 = \mathbf{83.6\ mm} \quad \textbf{Ans.}$$

A solid aluminium shaft 1 m long and of 50 mm diameter is to be replaced by a hollow shaft of the same length and same outside diameter, so that the hollow shaft could carry the same torque and has the same angle of twist. What must be the inner diameter of the hollow shaft ?

Take modulus of rigidity for the aluminium as 28 GPa and that for steel as 85 GPa.

SOLUTION. Given: Length of aluminium shaft $(l_A) = 1\text{ m} = 1 \times 10^3$ mm ; Diameter of aluminium shaft $(D_A) = 50$ mm ; Length of steel shaft $(l_S) = 1\text{ m} = 1 \times 10^3$ mm ; Outside diameter of steel shaft $(D_S) = 50$ mm; Modulus of rigidity for aluminium $(C_A) = 28\text{ GPa} = 28 \times 10^3\text{ N/mm}^2$ and modulus of rigidity for steel $= 85\text{ GPa} = 85 \times 10^3\text{ N/mm}^2$.

Let d_S = Inner diameter of steel shaft in mm.

We know that polar moment of inertia of the solid aluminium shaft,

$$J_A = \frac{\pi}{32} \times D^4 = \frac{\pi}{32} \times (50)^4 \text{ mm}^4$$

We also know that relation for angle of twist

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \theta = \frac{T \cdot l}{JC}$$

$$\therefore \quad \theta_A = \frac{T_A \cdot l_A}{\frac{\pi}{32} \times (50)^4 \times 28 \times 10^3} \text{ rad.}$$

and $$\theta_S = \frac{T_S \cdot l_S}{\frac{\pi}{32} \times [(50)^4 - (d)^4] \times 85 \times 10^3} \text{ rad.}$$

Since both the angles of twists (*i.e.*, θ_A and θ_B) are same, therefore equating these values,

$$\frac{T_A \cdot l_A}{\frac{\pi}{32} \times (50)^4 \times 28 \times 10^3} = \frac{T_S \cdot l_S}{\frac{\pi}{32} \times [(50)^4 - (d)^4] \times 85 \times 10^3}$$

Substituting $T_A = T_S$ and $l_A = l_S$ in the above equation,

$$(50)^4 \times 28 = [(50)^4 - d^4] \times 85$$

$$175 \times 10^6 = (531.25 \times 10^6) - 85\, d^4$$

$$85\, d^4 = (531.25 \times 10^6) - (175 \times 10^6) = 356.25 \times 10^6$$

$$d^4 = \frac{356.25 \times 10^6}{85} = 4.191 \times 10^6 \text{ mm}^4$$

$\therefore$ d = **45.25 mm** **Ans.**

A hollow steel shaft of 300 mm external diameter and 200 mm internal diameter has to be replaced by a solid alloy shaft. Assuming the same values of polar modulus for both, calculate the diameter of the latter and work out the ratio of their torsional rigidities. Take C for steel as 2.4 C for alloy.

SOLUTION. Given: External diameter of steel shaft (D) = 300 mm ; Internal diameter of steel shaft (d) = 200 mm and modulus of rigidity for steel (C_S) = 2.4 (where C_A is the modulus of rigidity for the alloy).

Diameter of the solid alloy shaft

Let D_1 = Diameter of the solid alloy shaft.

We know that polar modulus of hollow steel shaft,

$$Z_S = \frac{\pi}{16D}(D^4 - d^4) = \frac{\pi}{16 \times 300}[(300)^4 - (200)^4] \text{ mm}^3$$

A hollow steel shaft of 300 mm external diameter and 200 mm internal diameter has to be replaced by a solid alloy shaft. Assuming the same values of polar modulus for both, calculate the diameter of the latter and work out the ratio of their torsional rigidities. Take C for steel as 2.4 C for alloy.

SOLUTION. Given: External diameter of steel shaft $(D) = 300$ mm ; Internal diameter of steel shaft $(d) = 200$ mm and modulus of rigidity for steel $(C_S) = 2.4$ (where C_A is the modulus of rigidity for the alloy).

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A hollow steel shaft of 300 mm external diameter and 200 mm internal diameter has to be replaced by a solid alloy shaft. Assuming the same values of polar modulus for both, calculate the diameter of the latter and work out the ratio of their torsional rigidities. Take C for steel as 2.4 C for alloy.

SOLUTION. Given: External diameter of steel shaft $(D) = 300$ mm ; Internal diameter of steel shaft $(d) = 200$ mm and modulus of rigidity for steel $(C_S) = 2.4$ (where C_A is the modulus of rigidity for the alloy).

Diameter of the solid alloy shaft

Let D_1 = Diameter of the solid alloy shaft.

We know that polar modulus of hollow steel shaft,

$$Z_S = \frac{\pi}{16D}(D^4 - d^4) = \frac{\pi}{16 \times 300}[(300)^4 - (200)^4] \text{ mm}^3$$

$$= \frac{8.125 \times 10^6 \pi}{6} \text{ mm}^3 \quad ...(i)$$

Similarly, polar modulus of solid alloy shaft,

$$Z_A = \frac{\pi}{16} D_1^3 \text{ mm}^2 \quad ...(ii)$$

Since the polar modulus for both the shafts are the same, therefore equating (*i*) and (*ii*),

$$\frac{8.125 \times 10^6 \pi}{6} = \frac{\pi}{16} D_1^3$$

or $$D_1^3 = \frac{8.125 \times 10^6 \times 16}{6} = 21.67 \times 10^6$$

∴ $$D_1 = \mathbf{278.8 \text{ mm}} \quad \textbf{Ans.}$$

Ratio of torsional rigidities

We know that the torsional rigidity of hollow steel shaft

$$= C_S \times J_S = 2.4\, C_A \times \frac{\pi}{32} [(300)^4 - (200)^4] \quad ...(iii)$$

Similarly, torsional rigidity for solid alloy shaft

$$= C_A \times J_A = C_A \times \frac{\pi}{32} \times D^4 = C_A \times \frac{\pi}{32} \times (278.8)^4 \quad ...(iv)$$

$$\therefore \quad \frac{\text{Torsional rigidity of hollow steel shaft}}{\text{Torsional rigidity of solid alloy shaft}}$$

$$= \frac{2.4\, C_A \times \frac{\pi}{32}[(300)^4 - (200)^4]}{C_A \times \frac{\pi}{32}(278.8)^4} = \mathbf{2.58} \qquad \textbf{Ans.}$$

Shaft of Varying Sections

Sometimes a shaft, made up of different lengths having different cross-sectional areas, is required to transmit some torque (or horse power) from one pulley to another.

A little consideration will show that for such a shaft, the torque transmitted by individual sections have to be calculated first and the minimum value of these torques will be the strength of such a shaft. The angle of twist for such a shaft may be found out as usual.

The stepped steel shaft shown in Fig. 27.4 is subjected to a torque (T) at the free end, and a torque (2T) in the opposite direction at the junction of the two sizes.

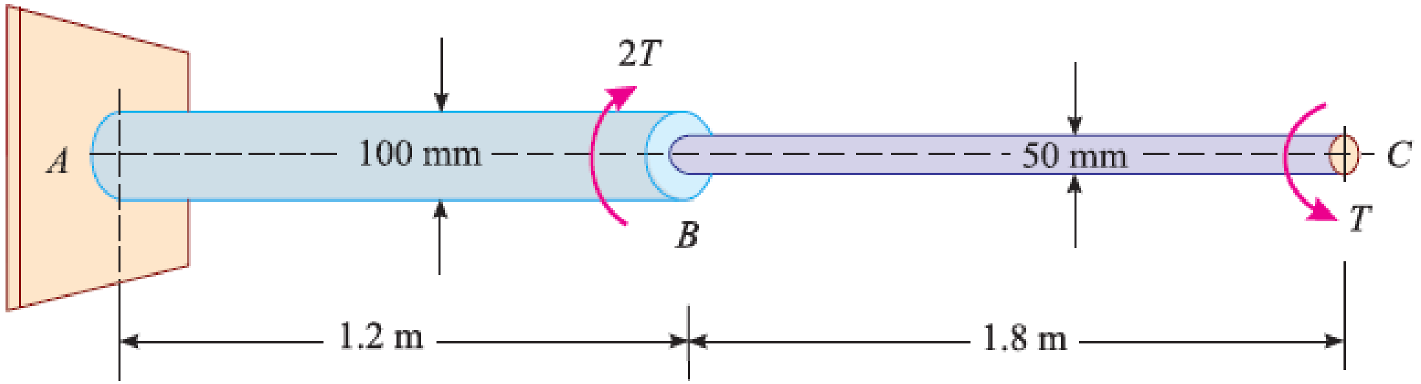


Fig. 27.4

What is the total angle of twist at the free end, if maximum shear stress in the shaft is limited to 70 MPa? Assume the modulus of rigidity to be 84 GPa.

SOLUTION. Given: Torque at $C = T$ (anticlockwise); Torque at $B = 2T$ (clockwise) ; Diameter of shaft AB (D_{AB}) = 100 mm ; Diameter of shaft BC (D_{BC}) = 50 mm ; Maximum shear stress (τ) = 70 MPa = 70 N/mm^2 and modulus of rigidity (C) = 84 GPa = 84×10^3 N/mm^2.

Since the torques at B and C are in opposite directions, therefore the effect of these two torques will be studied first independently, sum of the two twists (one in clockwise direction and the other in anticlockwise direction).

First of all, let us first find out the value of torque T at C. It may be noted that if the value of torque is obtained for the portion AB, it will induce more stress in the portion BC (because the portion BC is of less diameter). Therefore we shall calculate the torque for the portion BC (because it will not induce stress more than the permissible in the portion AB).

We know that the torque at C,

$$T = \frac{\pi}{16} \times \tau \times (D_{BC})^3 = \frac{\pi}{16} \times 70 \times (50)^3 = 1.718 \times 10^6 \text{ N-mm}$$

We also know that polar moment of inertia of the solid circular shaft AB,

$$J_{AB} = \frac{\pi}{32} \times (D_{AB})^4 = \frac{\pi}{32} \times (100)^4 = 9.82 \times 10^6 \text{ mm}^4$$

Similarly,

$$J_{BC} = \frac{\pi}{32} \times (D_{BC})^4 = \frac{\pi}{32} \times (50)^4 = 0.614 \times 10^6 \text{ mm}^4$$

$\therefore$ Angle of twist at C due to torque (T) at C,

$$\theta = \frac{T \cdot l}{J \cdot C} = \frac{T}{C}\left(\frac{l_{AB}}{J_{AB}} + \frac{l_{BC}}{J_{BC}}\right)$$

$$= \frac{1.718 \times 10^6}{84 \times 10^3}\left(\frac{1.2 \times 10^3}{9.82 \times 10^6} + \frac{1.8 \times 10^3}{0.614 \times 10^6}\right) \text{ rad}$$

$$= 20.45 \times (30.54 \times 10^{-4}) = 0.0624 \text{ rad} \qquad ...(i)$$

Similarly, angle of twist at C due to torque ($2T$) at B,

$$\theta = \frac{2T}{C} \times \frac{l_{AB}}{J_{AB}} = \frac{2 \times (1.718 \times 10^6)}{84 \times 10^3} \times \frac{1.2 \times 10^3}{9.82 \times 10^6} \text{ rad}$$

$$= 40.9 \times (1.222 \times 10^{-4}) = 0.005 \text{ rad} \qquad \text{...(ii)}$$

From the geometry of the shaft, we find that the twist at B (due to torque of $2T$ at B) will continue at C also. Since the directions of both the twists are opposite to each other, therefore net angle of twist at C

$$= 0.0624 - 0.005 = 0.0574 \text{ rad} = \mathbf{3.29°} \qquad \textbf{Ans.}$$

A shaft ABC of 500 mm length and 40 mm external diameter is bored, for a part of its length AB to a 20 mm diameter and for the remaining length BC to a 30 mm diameter bore as shown in Fig. 27.5. If the shear stress is not to exceed 80 MPa, find the maximum power, the shaft can transmit at a speed of 200 r.p.m.

If the angle of twist in the length of 20 mm diameter bore is equal to that in the 30 mm diameter bore, find the length of the shaft that has been bored to 20 mm and 30 mm diameter.

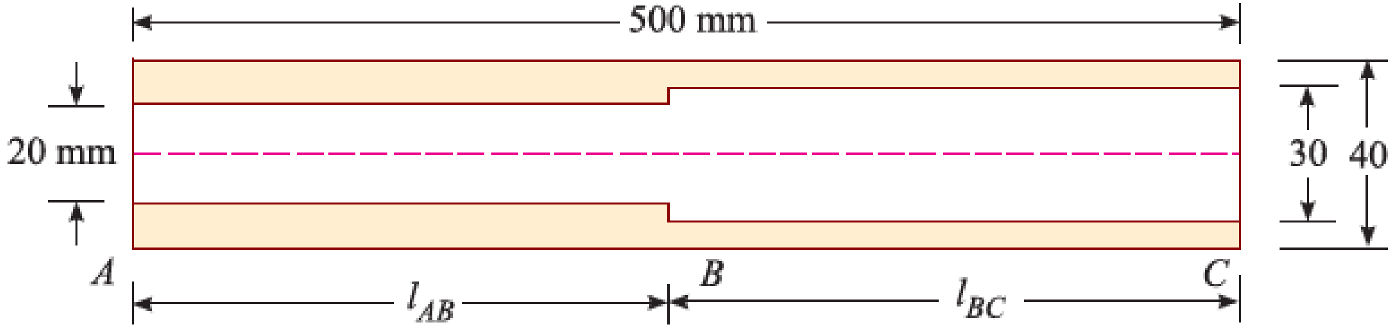


Fig. 27.5

SOLUTION. Given: Total length of the shaft (l) = 500 mm; External diameter of the shaft (D) = 40 mm ; Internal diameter of shaft AB (d_{AB}) = 20 mm ; Internal diameter of shaft BC (d_{BC}) = 30 mm ; Maximum shear stress (τ) = 80 MPa = 80 N/mm^2 and speed of the shaft (N) = 200 r.p.m.

Maximum power the shaft can transmit

We know that torque transmitted by the shaft AB,

$$T_{AB} = \frac{\pi}{16}\times\tau\times\left(\frac{D^4 - d_{AB}^4}{D}\right) = \frac{\pi}{16}\times 80\times\left[\frac{(40)^4 - (20)^4}{40}\right] \text{ N-mm}$$

$$= 942.5 \times 10^3 \text{ N-mm} \qquad ...(i)$$

Similarly,

$$T_{BC} = \frac{\pi}{16}\times\tau\times\left(\frac{D^4 - d_{BC}^4}{D}\right) = \frac{\pi}{16}\times 80\times\left[\frac{(40)^4 - (30)^4}{40}\right] \text{ N-mm}$$

$$= 687.3 \times 10^3 \text{ N-mm} \qquad ...(ii)$$

From the above two values, we see that the safe torque transmitted by the shaft is minimum of the two, *i.e.*, 687.3×10^3 N-mm = 687.3 N-m. Therefore maximum power the shaft can transmit,

$$P = \frac{2\pi NT}{60} = \frac{2\times\pi\times 200\times(687.3)}{60} = 14\ 394 \text{ W}$$

$$= \mathbf{14.39\ kW} \qquad \textbf{Ans.}$$

Length of the shaft, that has been bored to 20 mm diameter

Let l_{AB} = Length of the shaft *AB* (*i.e.*, 20 mm diameter bore) and

l_{BC} = Length of the shaft *BC* (*i.e.*, 30 mm diameter bore) equal to $(500 - l_{AB})$ mm.

We know that polar moment of inertia for the shaft *AB*,

$$J_{AB} = \frac{\pi}{32}\times(D^4 - d_{AB}^4) = \frac{\pi}{32}\times[(40)^4 - (20)^4]\ \text{mm}^4$$

Similarly,
$$J_{BC} = \frac{\pi}{32}\times(D^4 - d_{BC}^4) = \frac{\pi}{32}\times[(40)^4 - (30)^4]\ \text{mm}^4$$

We know that relation for the angle of twist:

$$\frac{T}{J} = \frac{C\,\theta}{l} \quad \text{or} \quad \theta = \frac{T\cdot l}{JC}$$

$$\therefore \quad \theta_{AB} = \frac{T\cdot l_{AB}}{J_{AB}\cdot C} \quad \text{and} \quad \theta_{BC} = \frac{T\cdot l_{BC}}{J_{BC}\cdot C}$$

Since $\theta_{AB} = \theta_{AC}$ and T as well as C is equal in both these cases, therefore

$$\frac{l_{AB}}{J_{AB}} = \frac{l_{BC}}{J_{BC}} \quad \text{or} \quad \frac{l_{AB}}{\frac{\pi}{32}\times[(40)^4 - (20)^4]} = \frac{l_{BC}}{\frac{\pi}{32}\times[(40)^4 - (30)^4]}$$

or
$$\frac{l_{AB}}{l_{BC}} = \frac{(40)^4 - (20)^4}{(40)^4 - (30)^4} = \frac{2400000}{1750000} = 1.37$$

$$\therefore \quad l_{AB} = 1.37\, l_{BC}$$

$$1.37\, l_{BC} + l_{BC} = 500 \qquad \ldots(\because l_{AB} + l_{BC} = 500)$$

$$\therefore \quad l_{BC} = \frac{500}{2.37} = \mathbf{211\ mm} \qquad \textbf{Ans.}$$

and
$$l_{AB} = 500 - 211 = \mathbf{289\ mm} \qquad \textbf{Ans.}$$

Composite Shaft

Sometimes, a shaft is made up of composite section *i.e.*, one type of shaft rigidly sleeved over another type of shaft. At the time of sleeving, the two shafts are joined together in such a way, that the composite shaft behaves like a single shaft. The total torque transmitted by the composite shaft is shared by the two shafts, depending upon their diameters and elastic properties.

A composite shaft consists of copper rod of 30 mm diameter enclosed in a steel tube of external diameter 40 mm and 5 mm thick. The shaft is required to transmit a torque of 0.5 kN-m. Determine the shearing stresses developed in the copper and steel, if both the shafts have equal lengths and welded to a plate at each end, so that their twists are equal. Take C_C = 40 GPa and C_S = 80 GPa.

SOLUTION. Given: Diameter of copper rod (D_C) = 30 mm ; External diameter of steel tube (D_S) = 40 mm ; Thickness of steel tube = 5 mm ; Therefore internal diameter of steel tube (d_S) = 40 – (2 × 5) = 30 mm; Total torque to be transmitted (T) = 0.5 kN-m = 0.5×10^6 N-mm ; Modulus of rigidity for copper (C_C) = 40 GPa = 40×10^3 N/mm² and modulus of rigidity for steel (C_S) = 80 GPa = 80×10^3 N/mm².

Let

T_C = Torque shared by copper rod,

τ_C = Shear stress developed in the copper rod and

T_S, τ_S = Corresponding values for steel tube.

∴ Total torque (T)

$$T_C + T_S = 0.5 \times 10^6 \text{ N-mm} \quad ...(i)$$

We know that polar moment of inertia of copper rod,

$$J_C = \frac{\pi}{32} \times (D_C^4) = \frac{\pi}{32} \times (30)^4 = \frac{0.81 \times 10^6\ \pi}{32} \text{ mm}^4$$

and polar moment of inertia of steel tube,

$$J_S = \frac{\pi}{32} \times (D_S^4 - d_S^4) = \frac{\pi}{32}[(40)^4 - (30)^4] = \frac{1.75 \times 10^6\ \pi}{32}\ \text{mm}^4$$

We also know that relation for angle of twist:

$$\frac{T}{J} = \frac{C\,\theta}{l} \qquad \text{or} \qquad \theta = \frac{T \cdot l}{JC}$$

$$\therefore \qquad \theta_C = \frac{T_C \cdot l}{J_C \cdot C_C} = \frac{T_C \cdot l}{\frac{0.81 \times 10^6\ \pi}{32} \times (40 \times 10^3)} = \frac{T_C \cdot l}{1012.5 \times 10^6\ \pi}\ \text{rad.}$$

Similarly,

$$\theta_S = \frac{T_S \cdot l}{J_S \cdot C_S} = \frac{T_S \cdot l}{\frac{1.75 \times 10^6\ \pi}{32} \times (80 \times 10^3)} = \frac{T_S \cdot l}{4375 \times 10^6\ \pi}\ \text{rad.}$$

Since θ_C is equal to θ_S, therefore equating these values,

$$\frac{T_C \cdot l}{1012.5 \times 10^6\ \pi} = \frac{T_S \cdot l}{4375 \times 10^6\ \pi} \qquad \text{or} \qquad T_C = \frac{81\,T_S}{350}$$

Substituting this value of T_C in equation (*i*),

$$\frac{81\,T_S}{350} + T_S = 0.5 \times 10^6 \quad \text{or} \quad \frac{431\,T_S}{350} = 0.5 \times 10^6$$

$$\therefore \quad T_S = \frac{(0.5 \times 10^6) \times 350}{431} = 0.406 \times 10^6 \text{ N-mm}$$

and

$$T_C = \frac{81\,T_S}{350} = \frac{81 \times (0.406 \times 10^6)}{350}$$

$$= \mathbf{0.094 \times 10^6 \text{ N-mm}} \quad \textbf{Ans.}$$

We know that torque transmitted by copper rod (T_C),

$$0.094 \times 10^6 = \frac{\pi}{16} \times \tau_C \times D_C^3 = \frac{\pi}{16} \times \tau_C \times (30)^3 = 5301\,\tau_C$$

$$\therefore \quad \tau_C = \frac{0.094 \times 10^6}{5301} = 17.7 \text{ N/mm}^2 = \mathbf{17.7 \text{ MPa}} \quad \textbf{Ans.}$$

Similarly, torque transmitted by steel tube (T_S),

$$0.406 \times 10^6 = \frac{\pi}{16} \times \tau_S \times \left(\frac{D_S^4 - d_S^4}{D_S}\right) = \frac{\pi}{16} \times \tau_S \times \left(\frac{(40)^4 - (30)^4}{40}\right) = 8590\,\tau_S$$

$$\therefore \quad \tau_S = \frac{0.406 \times 10^6}{8590} = 47.3 \text{ N/mm}^2 = \mathbf{47.3 \text{ MPa}} \quad \textbf{Ans.}$$

A composite shaft consists of a steel rod of 60 mm diameter surrounded by a closely fitting tube of brass. Find the outside diameter of the brass tube, when a torque of 1 kN-m is applied on the composite shaft and shared equally by the two materials. Take C for steel as 84 GPa and C for brass as 42 GPa.

Also determine the common angle of twist in a length of 4 metres.

SOLUTION. Given: Diameter of steel rod $(D_S) = 60$ mm ; Inner diameter of brass tube $(d_S) = 60$ mm ; Total torque $(T) = 1$ kN-m $= 1 \times 10^6$ N-mm ; Torque shared by steel rod $(T_S) = \frac{1}{2} \times (1 \times 10^6) = 0.5 \times 10^6$ N-mm ; Torque shared by brass tube $(T_B) = (1 \times 10^6) - (0.5 \times 10^6) = 0.5 \times 10^6$ N-mm ; Modulus of rigidity for steel $(C_S) = 84$ GPa $= 84 \times 10^3$ N/mm^2 ; Modulus of rigidity for brass $(\sigma_B) = 42$ GPa $= 42 \times 10^3$ N/mm^2 and length of shaft $(l) = 4$ m $= 4 \times 10^3$ mm.

Outside diameter of the brass tube

Let D_B = Outside diameter of the brass tube in mm.

We know that polar moment of inertia of steel rod,

$$J_S = \frac{\pi}{32} \times (D_S)^4 = \frac{\pi}{32} \times (60)^4 = \frac{12.96 \times 10^6 \, \pi}{32} \text{ mm}^4$$

and polar moment of inertia of brass tube,

$$J_B = \frac{\pi}{32} \times (D_B^4 - d_B^4) = \frac{\pi}{32} \times [D_B^4 - (60)^4] \text{ mm}^4$$

We also know that the relation for angle of twist:

$$\frac{T}{J} = \frac{C\,\theta}{l} \quad \text{or} \quad \theta = \frac{T \cdot l}{JC}$$

$$\therefore \quad \theta_S = \frac{T \cdot l}{J_S \cdot C_S} = \frac{T \cdot l}{\frac{\pi}{32} \times (60)^4 \times (84 \times 10^3)} \qquad ...(i)$$

Similarly, $$\theta_B = \frac{T \cdot l}{J_B \cdot C_B} = \frac{T \cdot l}{\frac{\pi}{32} \times [D_B^4 - (60)^4] \times 42 \times 10^3} \quad \ldots(ii)$$

Since θ_S is equal to θ_B, therefore equating these two values,

$$\frac{T \cdot l}{\frac{\pi}{32} \times (60)^4 \times (84 \times 10^3)} = \frac{T \cdot l}{\frac{\pi}{32} \times [D_B^4 - (60)^4] \times 42 \times 10^3}$$

$$D_B^4 - 12.96 \times 10^6 = 2 \times (12.96 \times 10^6) = 25.92 \times 10^6$$

or $$D_B^4 = (25.92 \times 10^6) + (12.96 \times 10^6) = 38.88 \times 10^6 \text{ mm}^4$$

$\therefore$ D_B = **79 mm** **Ans.**

Common angle of twist

Let θ = Common angle of twist.

Substituting the values of T (equal to 0.5×10^6 N-mm) and l (equal to 4×10^3 mm) in equation (*i*),

$$\theta = \frac{(0.5 \times 10^6) \times (4 \times 10^3)}{\frac{\pi}{32} \times (60)^4 \times (84 \times 10^3)} = 0.0187 \text{ rad} = \mathbf{1.07^\circ}$$ **Ans.**

Buckling of columns: Introduction to buckling of column with its application, Different column conditions and critical, safe load determination by Euler's theory. Limitation of Euler's theory

Introduction

A structural member, subjected to an axial compressive force, is called a strut. As per definition, a strut may be horizontal, inclined or even vertical. But a vertical strut, used in buildings or frames, is called a *column*.

Axial Compression-Columns

A column may fail by buckling rather than by compression

- <u>Slenderness ratio</u> : (S_r)
- A **short column** will fail in compression.
- An **intermediate** or a **long column** will fail by buckling when the applied axial load exceeds some critical value.
- Compressive Stress can be well below the materials strength at the time of buckling.

The factor that determines if a column is short or long is its **Slenderness ratio.**

$S_r = \frac{l}{k}$ ____ k = radius of gyration

$k = \sqrt{\frac{I}{A}}$ ____ I = Smallest area moment of cross section inertia about neutral axis

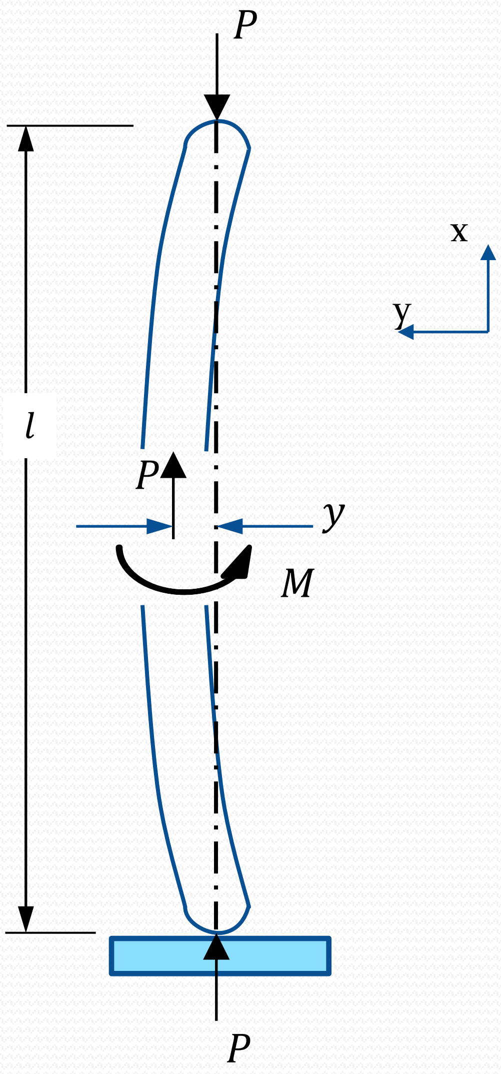
❑Short Columns:

- Usually Slenderness ratio < 10 (Different for different material)
- Fails in compression, hence materials yield strength in compression is used as the limiting factor for failure.

❑Long Columns:

Usually Slenderness ratio > 120 (Different for different material)

A long column requires calculation of critical load



Bending Moment , $M = Py$ ____(1)

For small deflections of beam ,

$\frac{M}{EI} = \frac{d^2y}{dx^2}$ ____(2)

From (1) & (2),

$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$ ____(3)

Solution of differential equation (3) is,

$$y = A sin\sqrt{\frac{P}{EI}}x + B cos\sqrt{\frac{P}{EI}}x$$

Boundary conditions for the column:

At $x = 0, y = 0$; $x = l, y = 0$

Substituting Boundary conditions , we get

$B = 0$ and $A sin\sqrt{\frac{P}{EI}}l = 0$

Since $A \neq 0,$ $\therefore sin\sqrt{\frac{P}{EI}}l = 0$

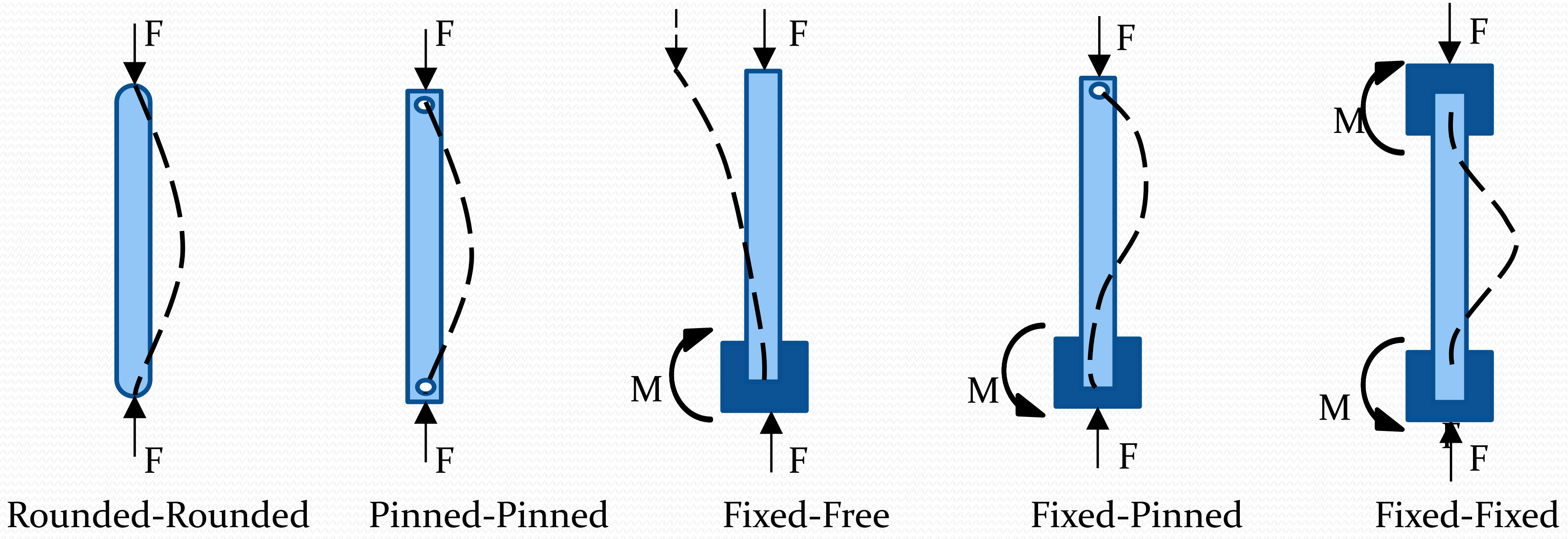
which gives, $\sqrt{\frac{P}{EI}}l = n\pi \; ; \; n = 1,2,3, \dots .$

The first critical load will occur at $n = 1,$

$$\therefore P_{cr} = \frac{\pi^2 EI}{l^2}$$

Above formula is as **Euler column formula**.

End Conditions for columns



End Conditions	Rounded-Rounded	Pinned-Pinned	Fixed-Free	Fixed-Pinned	Fixed-Fixed
Theoretical Value	$l_{eff} = l$	$l_{eff} = l$	$l_{eff} = 2l$	$l_{eff} = \frac{l}{\sqrt{2}}$	$l_{eff} = \frac{l}{2}$
AISC*Recommended	$l_{eff} = l$	$l_{eff} = l$	$l_{eff} = 2.1l$	$l_{eff} = 0.80l$	$l_{eff} = 0.65l$

*The American Institute of Steel Construction

A steel rod 5 m long and of 40 mm diameter is used as a column, with on end fixed and the other free. Determine the crippling load by Euler's formula. Take E as 200 GPa.

SOLUTION. Given : Length (l) = 5 m = 5×10^3 mm ; Diameter of column (d) = 40 mm and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2.

We know that moment of iertia of the column section,

$$I = \frac{\pi}{64} \times (d)^4 = \frac{\pi}{64} \times (40)^4 = 40\,000\,\pi \text{ mm}^4$$

Since the column is fixed at one end and free at the other, therefore equivalent length of the column,

$$L_e = 2\,l = 2 \times (5 \times 10^3) = 10 \times 10^3 \text{ mm}$$

$\therefore$ Euler's crippling load,

$$P_E = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times (200 \times 10^3) \times (40000\,\pi)}{(10 \times 10^3)^2} = 2480 \text{ N}$$

$$= 2.48 \text{ kN} \qquad \textbf{Ans.}$$

A hollow alloy tube 4 m long with external and internal diameters of 40 mm and 25 mm respectively was found to extend 4.8 mm under a tensile load of 60 kN. Find the buckling load for the tube with both ends pinned. Also find the safe load on the tube, taking a factor of safety as 5.

SOLUTION. Given : Length $l = 4$ m ; External diameter of column $(D) = 40$ mm ; Internal diameter of column $(d) = 25$ mm ; Deflection $(\delta l) = 4.8$ mm ; Tensile load = 60 kN = 60×10^3 N and factor of safety = 5.

Buckling load for the tube

We know that area of the tube,

$$A = \frac{\pi}{4} \times [D^2 - d^2] = \frac{\pi}{4}[(40)^2 - (25)^2] = 765.8 \text{ mm}^2$$

and moment of inertia of the tube,

$$I = \frac{\pi}{64}[D^4 = d^4] = \frac{\pi}{64}[(40)^4 - (25)^4] = 106\ 500 \text{ mm}^4$$

We also know that strain in the alloy tube,

$$e = \frac{\delta l}{l} = \frac{4.8}{4 \times 10^3} = 0.0012$$

and modulus of elasticity for the alloy,

$$E = \frac{\text{Load}}{\text{Area} \times \text{Strain}} = \frac{60 \times 10^3}{765.8 \times 0.0012} = 65\ 290 \text{ N/mm}^2$$

Since the column is pinned at its both ends, therefore equivalent length of the column,

$$L_e = l = 4 \times 10^3 \text{ mm}$$

$$\therefore \quad \text{Euler's buckling load, } P_E = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times 65290 \times 106500}{(4 \times 10^3)^2} = 4290 \text{ N}$$

$$= 4.29 \text{ kN} \qquad \textbf{Ans.}$$

Safe load for the tube

We also know that safe load for the tube

$$= \frac{\text{Buckling load}}{\text{Factor of safety}} = \frac{4.29}{5} = 0.858 \text{ kN} \qquad \textbf{Ans.}$$

Compare the ratio of the strength of a solid steel column to that of a hollow of the same cross-sectional area. The internal diameter of the hollow column is 3/4 of the external diameter. Both the columns have the same length and are pinned at both ends.

SOLUTION. Give : Area of solid steel colum $A_S = A_H$ (where A_H = Area of hollow column) ; Internal diameter of hollow column $(d) = 3\ D/4$ (where D = External diameter) and length of solid column $(l_S) = l_H$ (where l_H = Length of hollow column).

Let D_1 = Diameter of the solid column,

k_H = Radius of gyration for hollow column and

k_S = Radius of gyration for solid column.

Since both the columns are pinned at their both ends, therefore equivalent length of the solid column,

$$L_S = l_S = L_H = l_H = L$$

We know that Euler's crippling load for the solid column,

$$P_S = \frac{\pi^2 EI}{L_H^2} = \frac{\pi^2 E \cdot A_S \cdot k_S^{\ 2}}{L^2} \qquad ...(i)$$

Similarly Euler's crippling load for the hollow column

$$P_H = \frac{\pi^2 EI}{L_H^2} = \frac{\pi^2 E \cdot A_H \cdot k_H{}^2}{L^2} \quad \ldots(ii)$$

Dividing equation (*ii*) by (*i*),

$$\frac{P_H}{P_S} = \left(\frac{k_H}{k_S}\right)^2 = \frac{\frac{D^2 + d^2}{16}}{\frac{D_1^2}{16}} = \frac{D^2 + d^2}{D_1^2} = \frac{D^2 = \left(\frac{3D}{4}\right)^2}{D_1^2}$$

$$= \frac{25\,D^2}{16\,D_1^2} \quad \ldots(iii)$$

Since the cross-sectional areas of the both the columns is equal, therefore

$$\frac{\pi}{4} \times D_1^2 = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}\left[D^2 - \left(\frac{3\,D}{4}\right)^2\right] = \frac{\pi}{4} \times \frac{7\,D^2}{16}$$

$$\therefore \quad D_1^2 = \frac{7\,D^2}{16}$$

Now substituting the value of D_1^2 in equation (*iii*),

$$\frac{P_H}{P_S} = \frac{25\,D^2}{16 \times \frac{7\,D^2}{16}} = \frac{25}{7} \qquad \textbf{Ans.}$$

An I section joist 400 mm × 200 mm × 20 mm and 6 m long is used as a strut with both ends fixed. What is Euler's cripp;ing load for the column? Take Young's modulus for the joist as 200 GPa.

SOLUTION. Given : Outer depth (D) = 400 mm ; Outer width (B) = 200 mm ; Length (l) = 6 m = 6×10^3 mm and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2.

From the geometry of the figure, we find that inner depth,

$$d = 400 - (2 \times 20) = 360 \text{ mm}$$

and inner width, $$b = 200 - 20 = 180 \text{ mm}$$

We know that moment of inertia of the joist section about X-X axis,

$$I_{XX} = \frac{1}{12}[BD^2 - ba^3]$$

$$= \frac{1}{12}[200 \times (400)^3 - 180 \times (360)^3] \text{mm}^4$$

$$= 366.8 \times 10^6 \text{ mm}^4 \quad ...(i)$$

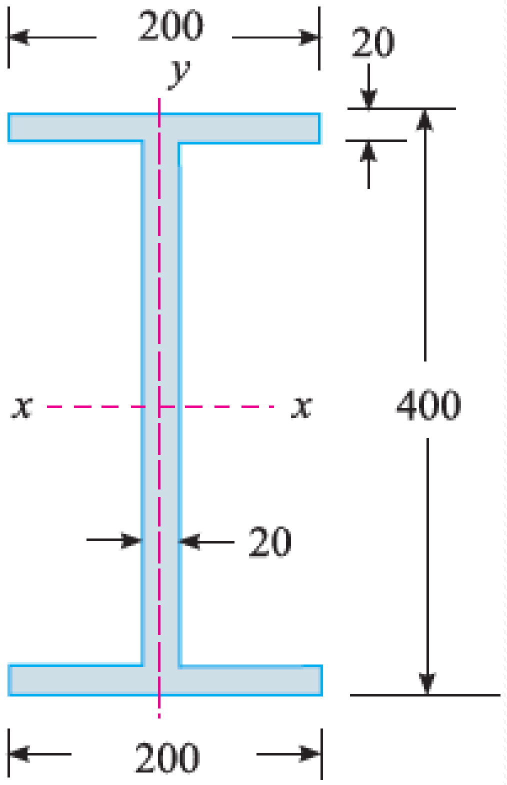


Fig. 34.6

Similarly, $$I_{YY} = \left[2 \times \frac{2 \times (200)^3}{12}\right] + \frac{360 \times (20)^3}{12} \text{mm}^4$$

$$= 2.91 \times 10^6 \text{ mm}^4 \quad ...(ii)$$

Since I_{YY} is less than I_{XX}, therefore the joist will tend to buckle in *Y-Y* direction. Thus, we shall take the value of *I* as $I_{YY} = 2.91 \times 10^6$ mm^4. Moreover, as the column is fixed at its both ends, therefore equivalent length of the column,

$$L_e = \frac{l}{2} = \frac{(6 \times 10^3)}{2} = 3 \times 10^3 \text{ mm}$$

∴ Euler's crippling load for the column,

$$P_E = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times (200 \times 10^3) \times (2.91 \times 10^6)}{(3 \times 10^3)^2} = 638.2 \times 10^3 \text{ N}$$

$$= 638.2 \text{ kN}$$ **Ans.**

A T-section 150 mm × 120 mm × 20 mm is used as a strut of 4 m long with hinged at its both ends. Calculate the crippling load, if Young's modulus for the material be 200 GPa.

SOLUTION. Given : Size of *T*-section = 150 mm × 120 mm × 20 mm ; Length (l) = 4 m = 4×10^3 mm and Young's modulus (E) = 200 GPa = 200×10^3 N/mm^2.

First of all, let us find the centre of the *T*-section; Let bottom of the web be the axis of reference.

Web

$$a_1 = 100 \times 20 = 2000 \text{ mm}^2$$

$$y_1 = \frac{100}{2} = 50 \text{ mm}$$

Flange

$$a_2 = 150 \times 20 = 3000 \text{ mm}^2$$

$$y_2 = 120 - \left(\frac{20}{2}\right) = 110 \text{ mm}$$

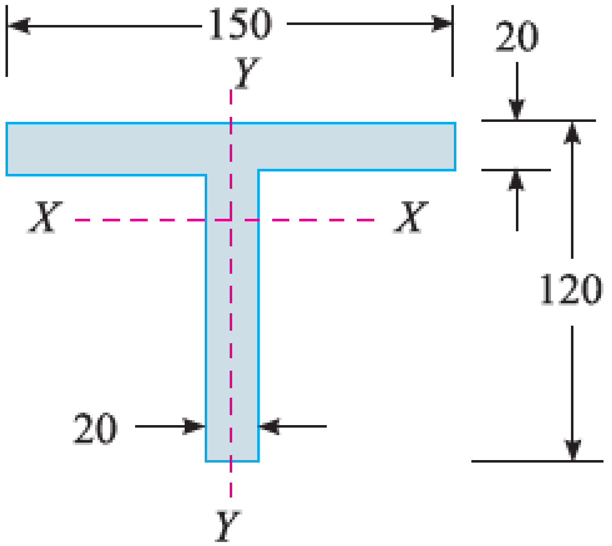


Fig. 34.7

We know that distance between the centre of gravity of the T-section and bottom of the web

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(2000 \times 50) + (3000 \times 110)}{200 + 3000} = 86 \text{ mm}$$

We also know that moment of inertia of the *T*-section about *X-X* axis,

$$I_{XX} = \left(\frac{20 \times (100)^3}{12} + 2000 \times (36)^2\right) + \left(\frac{150 \times (20)^3}{12} + 3000 \times (24)^2\right) \text{mm}^4$$

$$= (4.26 \times 10^6) + (1.83 \times 10^6) = 6.09 \times 10^6 \text{ mm}^4$$

Similarly,
$$I_{YY} = \frac{100 \times 9200)^3}{12} + \frac{20 \times (150)^3}{12} = 5.069 \times 10^6 \text{ mm}^4$$

Since I_{YY} is less than I_{XX}, therefore the column will tend to buckle in Y-Y direction. Thus, we shall take the value of I as $I_{YY} = 5.69 \times 10^6$ mm^4. Moreover, as the column is hinged at its both ends, therfore length of the column,

$$L_e = l = 4 \times 10^3 \text{ mm}$$

$$\therefore \text{ Euler's crippling load, } P_E = \frac{\pi^2 EI}{L_e^{\ 2}} = \frac{\pi^2 \times (200 \times 10^3) \times (5.69 \times 10^6)}{(4 \times 10^3)^2} = 702 \times 10^3 \text{ N}$$

$$= 702 \text{ kN}$$ **Ans.**

Limitation of Euler's Formula

We have discussed in Art. 32.12 that the Euler's formula for the crippling load,

$$P_E = \frac{\pi^2 EA}{\left(\frac{L_e}{k}\right)^2}$$

∴ Euler's crippling stress,

$$\sigma_E = \frac{P}{A} = \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2}$$

A little consideration will show that the crippling stress will be high, when the slenderness ratio is small. We know that the crippling stress for a column cannot be more than the crushing stress of the column material. It is thus obvious that the Euler's formula will give the value of crippling stress of the column (equal to the crushing stress of the column material) corresponding to the slenderness ratio. Now consider a mild steel column. We know that the crushing stress for the mild steel is 320 MPa or 320 N/m^2 and Young's modulus for the mild steel is 200 GPa or 200 × 10^3 N/mm^2.

Now equating the crippling stress to the crushing stress,

$$320 = \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2} = \frac{\pi^2 \times (200 \times 10^3)}{\left(\frac{L_e}{k}\right)^2}$$

Empirical Formulae for Columns

We have already discussed in the previous articles that the Euler's formula is valid only for long columns *i.e.*, for columns, whose slenderness ratio is greater than a certain value for a particular material. Moreover, it does not take into consideration the direct compressive stress. In order to fill up this lacuna, many more formulae were proposed by different scientists all over the world. The following empirical formulae, out of those, are important from the subject point of view.

1. Rankine's formula,
2. Johnson's formula, and
3. Indian Standard code.

Rankine's Formulae for Columns

We have already discussed that the Euler's formula gives correct results only for very long columns. Though this formula is applicable for columns, ranging from very long to short ones, yet it does not give reliable results. Prof. Rankine, after a number of experiments, gave the following empirical formula for columns.

$$\frac{1}{P_R} = \frac{1}{P_{CS}} + \frac{1}{P_E} \qquad ...(i)$$

where P_R = Crippling load by Rankine's Formula

$P_{CS} = \sigma_{CS} \cdot A$ = Ultimate crushing load for the column and

$P_E = \frac{\pi^2 EI}{L_e{}^2}$ = Crippling load obtained by Euler's formula.

A little consideration will show that the value of P_{CS} will remain constant irrespective of the fact whether the column is a long one or short one. Now, we shall study the effect of P_E in short as well as long columns one by one.

1. *Short columns.* In case of short columns, the value of P_E will be very high, therefore the value of $\frac{1}{P_E}$ will be quite negligible as compared to $\frac{1}{P_{CS}}$. It is thus obvious that the Rankine's formula will give the value of its crippling load (*i.e.*, *P*) approximately equal to the ultimate crushing load (*i.e.*,).

2. *Long columns.* In case of long columns, the value of P_E will be very small, therefore the value of $\frac{1}{P_E}$ will be quite considerable as compared to $\frac{1}{P_{CS}}$. It is thus obvious that the Rankine's formula will give the value of its crippling load (*i.e.*, *P*) approximately equal to the crippling load by Euler's formula (*i.e.*, P_E). Thus, we see that the Rankine's formula gives a fairly correct result for all cases of columns, ranging from short to long columns.

From equation (*i*), we know that

$$\frac{1}{P_R} = \frac{1}{P_{CS}} + \frac{1}{P_E} = \frac{P_E + P_{CS}}{P_{CS} \cdot P_E}$$

$$P_R = \frac{P_{CS} \cdot P_E}{P_{CS} + P_E} = \frac{P_{CS}}{1 + \frac{P_{CS}}{P_E}}$$

Now substituting the values of P_{CS} and P_E is the above equation

$$P_R = \frac{\sigma_{CS} \cdot A}{1 + \sigma_{CS} \cdot A \times \frac{L_e^2}{\pi^2 E}} = \frac{\sigma_{CS} \cdot A}{1 + \frac{\sigma_{CS}}{\pi^2 E} \times \frac{AL_e^2}{Ak^2}} \qquad ...(\because I = Ak^2)$$

or

$$P_R = \frac{\sigma_{CS} \cdot A}{1 + a\left(\frac{L_e}{k}\right)^2} = \frac{P_{CS}}{1 + a\left(\frac{L_e}{k}\right)^2}$$

where

P_{CS} = Crushing load of the column material

σ_{CS} = Crushing stress of the column material,

A = Cross-sectional area of the column,

a = Rankine's constant $\left(\text{equal to } \frac{\sigma_C}{\pi^2 E}\right)$

L_e = Equivalent length of the column, and

k = Least radius of gyration.

Find the Euler's crippling load for a hollow cylindrical steel column of 38 mm external diameter and 2.5 mm thick. Take length ofthe column as 2.3 m and hinged at its both ends. Take E = 205 GPa.

Also determine crippling load by Rankine's formula using constants as 335 MPa and $\frac{1}{7500}$

SOLUTION. Give : External diameter (D) = 38 mm ; Thickness = 2.5 mm or inner diameter (d) = 38 – (2 × 2.5) = 33 mm ; Length of the column (l) = 2.3 m = 2.3×10^3 mm ; Yield stress (σ_C) = 335 MPa = 335 N/mm² and Rankine's constant (a) = $\frac{1}{7500}$.

Euler's crippling load

We know that moment of inertia of the column section,

$$I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}[(38)^4 - (33)^4] = 14.05 \times 10^3 \pi \text{ mm}^4$$

Since the column is hinged at its both ends, therefore effective length of the column,

$$L_e = l = 2.3 \times 10^3 \text{ mm}$$

$\therefore$ Euler's crippling load, $P_E = \dfrac{\pi^2 EI}{L_e^{\,2}} = \dfrac{\pi^2 (205 \times 10^3) \times (14.05 \times 10^3\,\pi)}{(2.3 \times 10^3)^2} = 16\,880$ N

$= 16.88$ kN **Ans.**

Rankine's crippling load

We know that area of the column section,

$$A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}[(38)^2 - (33)^2] = 88.75\,\pi \text{ mm}^2$$

and least radius of gyration, $k = \sqrt{\dfrac{I}{A}} = \sqrt{\dfrac{14.05 \times 10^3\,\pi}{88.75\pi}} = 12.6$ mm

$\therefore$ Rankine's crippling load,

$$P_R = \frac{\sigma_{CS} \cdot A}{1 + a\left(\dfrac{L_e}{k}\right)^2} = \frac{335 \times 88.75\pi}{1 + \dfrac{1}{7500}\left(\dfrac{2.3 \times 10^3}{12.6}\right)^2} = 17\,160 \text{ N}$$

$= 17.16$ kN **Ans.**

Figure 34.8 shows a built-up column consisting of 150 mm × 100 mm R.S.J. with 120 mm × 12 mm plate riveted to each flange.

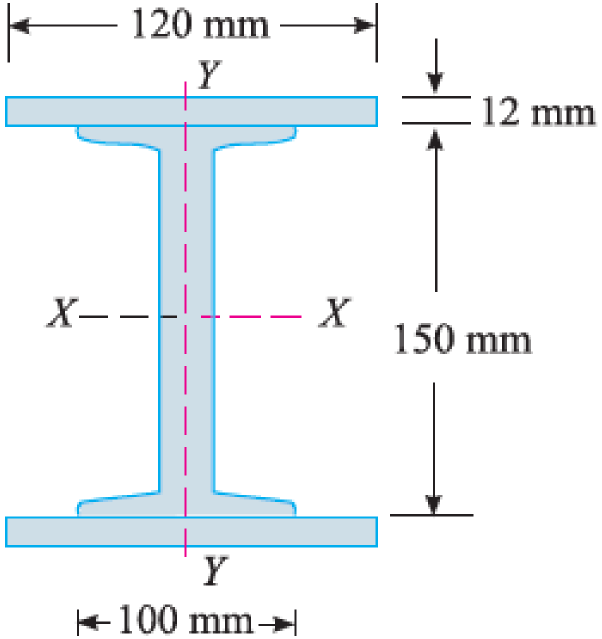


Fig. 34.8

Calculate the safe load, the column can carry, if it is 4 m long having one end fixed and the other hinged with a factor of safety 3.5. Take the properties of the joist as Area = 2167 mm², I_{XX} = 8.391 × 10⁶ mm⁴, I_{YY} = 0.948 × 10⁶ mm⁴. Assume the yield stress as 315 MPa and Rankine's constant (a) = $\frac{1}{7500}$.

SOLUTION. Given : Length of the column (l) = 4 m = 4×10^3 mm ; Factor of safety = 3.5 ; Yield stress (σ_C) = 315 MPa = 315 N/mm^2 ; Area of joist = 2167 mm^2 ; Moment of inertia, about *X-X* axis (I_{XX}) = 8.391×10^6 mm^4 ; Moment of inertia about *Y-Y* axis (I_{YY}) = 0.948×10^6 mm^4 and Rankine's

1

constant $(a) = \frac{1}{7500}$.

From the geometry of the figure, we find that area of the column section,

$$A = 2167 + (2 \times 120 \times 12) = 5047 \text{ mm}^2$$

and moment of inertia of the column section bout X-X axis,

$$I_{XX} = (83.91 \times 10^6) + 2\left[\frac{120 \times (12)^3}{12} + 120 \times 12 \times (81)^2\right] \text{mm}^4$$

$$= (8.391) \times 10^6) + (18.93 \times 10^6) = 27.32 \times 10^6 \text{ mm}^4$$

Similarly,

$$I_{YY} = (0.948 \times 10^6) + 2\left[\frac{12 \times (120)^3}{12}\right] \text{mm}^4$$

$$= (0.948 \times 10^6) + (3.456 \times 10^6) = 4.404 \times 10^6 \text{ mm}^4$$

Since I_{YY} is less than I_{XX}, therefore the column will tend to buckle in Y-Y direction. Thus we shall take I equal to $I_{YY} = 4.404 \times 10^6 \text{ mm}^4$ (*i.e.*, least of two). Moreover as the column is fixed at one end and hinged at the other, therefore equivalent length of the column.

$$L_e = \frac{l}{\sqrt{2}} = \frac{4 \times 10^3}{\sqrt{2}} = 2.83 \times 10^3 \text{ mm}$$

$$L_e = \frac{l}{\sqrt{2}} = \frac{4 \times 10^3}{\sqrt{2}} = 2.83 \times 10^3 \text{ mm}$$

We know that least radius of gyration,

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{4.404 \times 10^6}{5047}} = 29.5 \text{ mm}$$

∴ Rankine's crippling load on the column

$$P_R = \frac{\sigma_C \cdot A}{1 + a\left(\frac{L_e}{k}\right)^2} = \frac{315 \times 5047}{1 + \frac{1}{7500}\left(\frac{2.83 \times 10^3}{29.5}\right)^2}$$

$$= 714 \times 10^3 \text{ N} = 714 \text{ kN}$$

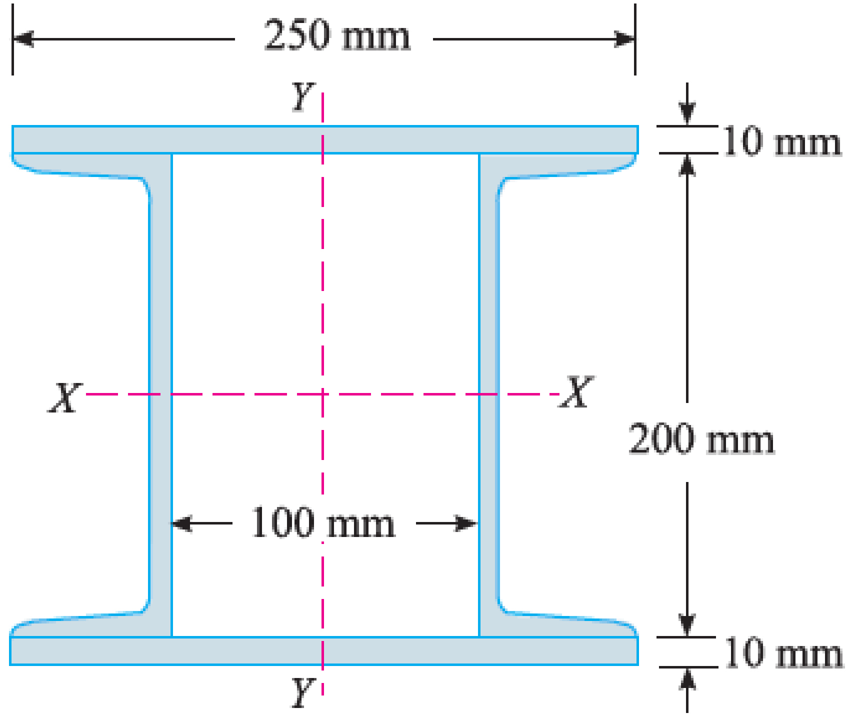
and safe load on the column

$$= \frac{\text{Crippling load}}{\text{Factor of safety}} = \frac{714}{3.5} = 204 \text{ kN} \qquad \textbf{Ans.}$$

A column is made up of two channels. ISJC 200 and two 250 mm × 10 mm flange plates as shown in Fig. 34.9.

Determine by Rankine's formula the safe load, the column of 6 m length, with both ends fixed, can carry with a factor of safety 4. The properties of one channel are Area = 1777 mm^2, I_{XX} = 11.612×10^6 mm^4 and $I_{YY} = 0.842 \times 10^6$ mm^4. Distance of centroid from back to web = 19.7 mm.

Take σ_C = 320 MPa and Rankine's constant = $\frac{1}{7500}$



Solution. Given : Length of the column (l) = 6 m = 6×10^3 mm ; Factor of safety = 4 ; Area of channel = 1777 mm^2 ; Moment of inertia about X-X axis (I_{XX}) = 11.612×10^6 mm^4 ; Moment of inertia about y-y axis (I_{YY}) = 0.842×10^6 mm^4 ; Distance of centroid from the back of web = 19.7 mm; Crushing stress (σ_C) = 320 MPa = 320 N/mm^2 and Rankine's constant (a) = $\frac{1}{7500}$.

From the geometry of the figure, we find that area of trhe column section,

$$A = 2\,[1777 + (250 \times 10)] = 8554 \text{ mm}^2$$

and moment of inertia of the column section about X-X axis,

$$I_{XX} = (2 \times 11.612 \times 10^6) + 2\left[\frac{250 \times (10)^2}{12} + (250 \times 10) \times (105)^2\right] \text{mm}^4$$

$$= (23.224 \times 10^6) + (55.167 \times 10^6) = 78.391 \times 10^5 \text{ mm}^4$$

Similarly,

$$I_{YY} = 2\left[\frac{10 \times (250)^3}{12} = (0.842 \times 10^6) + 1777 \times (50 + 19.7)^2\right] \text{mm}^4$$

$$= 2\,[13.021 \times 10^6) + (9.475 \times 10^6)] = 44.992 \times 10^6 \text{ mm}^4$$

Since L_{YY} is less than I_{XX}, therefore the column will tend to buckle in Y-Y direction. Thus we shall take I equal to $I_{YY} = 44.992 \times 10^6$ mm^4 (*i.e.*, least of the two). Moreover as the column is fixed at its both ends, therefore equivalent length of the colum,

$$L_e = \frac{l}{2} = \frac{6 \times 10^3}{2} = 3 \times 10^3 \text{ mm}$$

We know that least radius of gyration,

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{44.992 \times 10^6}{8554}} = 72.5 \text{ mm}$$

∴ Rankine's crippling load on the column,

$$P_R = \frac{\sigma_C \cdot A}{1 + a\left(\frac{L_e}{k}\right)^2} = \frac{320 \times 8554}{1 + \frac{1}{7500} \times \left(\frac{3 \times 10^3}{72.5}\right)^2}$$

$$= 2228.5 \times 10^3 \text{ N} = 2228.5 \text{ kN}$$

and Safe load on the column $= \frac{\text{Crippling load}}{\text{Factor of safety}} = \frac{2228.5}{4} = 557.1 \text{ kN}$ **Ans.**